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Numerical Approximation of Periodic Solutions of van der Pol's Equation

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Two new discrete methods, one based on discrete mechanics, the other based on high-order Taylor series, are developed and applied to approximate periodic solutions of van der Pol's equation. Typical numerical results are exhibited from a broad spectrum of values of λ in the range of physical interest.

1. INTRODUCTION

Though periodic solutions of nonlinear differential equations have long been of interest in applied science, the application of classical mathematical techniques has limited inquiry largely to questions relating to existence, uniqueness, and asymptotic estimates (see, e.g., Refs. [3, 4, 7, 9] and the additional references contained therein). With the development of the digital computer, both discrete and continuous numerical techniques were devised for approximating such periodic functions [2, 6, 10–16]. The discrete methods are based largely on Newton's method and Runge–Kutta formulas, while the continuous methods depend primarily on Tchebychev interpolation and Galerkin approximation.

In this paper we will develop two new methods for approximating periodic solutions of the van der Pol equation

$$\ddot{x} - \lambda(1 - x^2)\dot{x} + x = 0, \qquad \lambda > 0, \tag{1.1}$$

subject to the initial condition

$$\dot{x}(0) = 0. \tag{1.2}$$

Since, for each value of λ, it is known that there exists a unique periodic solution of (1.1)–(1.2), our problem is, in fact, that of approximating

$$x(0) = a \tag{1.3}$$

so that the initial value problem (1.1)–(1.3) has a periodic solution. In this connection, it will be convenient to let $T = T(\lambda)$ represent the period of the solution and to note that

$$x(T/2) = -a, \tag{1.4}$$

$$\dot{x}(T/2) = 0. \tag{1.5}$$

Each of the methods to be described is discrete, efficient, and self-starting in the sense that no knowledge about $x(0)$ is necessary *a priori* to assure convergence.

2. Method I: The Method of Discrete Mechanics

Let a particle be in motion in a fixed, say X, direction. For $\Delta t > 0$, let $t_k = k\Delta t$, $k = 0, 1,...,$ and let the particle be at x_k at time t_k. Defining particle velocity $v_k = v(t_k)$, $k = 1, 2,...,$ and particle acceleration $a_k = a(t_k)$, $k = 1, 2,...,$ by

$$\frac{v_k + v_{k-1}}{2} = \frac{x_k - x_{k-1}}{\Delta t}, \qquad k = 1, 2,..., \tag{2.1}$$

$$a_k = \frac{v_k - v_{k-1}}{\Delta t}, \qquad k = 1, 2,..., \tag{2.2}$$

and introducing a Newtonian dynamical equation in the discrete form

$$ma_{k+1} = F(x_k, t_k, v_k) \tag{2.3}$$

results in a discrete model of mechanics in which all the usual conservation laws are valid [5]. Further, every initial value problem for (2.3) has a unique discrete solution which can be generated recursively on a computer.

In the above context, then, let us choose a particle of unit mass and rewrite (1.1) in the form (2.3) to yield

$$a_{k+1} = \lambda(1 - x_k^2)\, v_k - x_k. \tag{2.4}$$

Since $\dot{x}(0) = v_0 = 0$, it follows from (2.1) and (2.2) that

$$v_1 = \frac{2}{\Delta t}(x_1 - x_0),$$

$$v_n = \frac{2}{\Delta t}\left\{x_n + (-1)^n x_0 + 2\sum_{j=1}^{n-1}[(-1)^j x_{n-j}]\right\}, \qquad n \geqslant 2,$$

$$a_1 = \frac{2}{(\Delta t)^2}[x_1 - x_0],$$

$$a_2 = \frac{2}{(\Delta t)^2}[x_2 - 3x_1 + 2x_0],$$

$$a_n = \frac{2}{(\Delta t)^2}\left\{x_n - 3x_{n-1} + 2(-1)^n x_0 + 4\sum_{j=2}^{n-1}[(-1)^j x_{n-j}]\right\}, \qquad n \geqslant 3,$$

which, upon substitution into (2.4), yields

$$x_1 = \left[1 - \frac{(\Delta t)^2}{2}\right] x_0 , \tag{2.5}$$

$$x_2 = x_1 \left[3 - \frac{(\Delta t)^2}{2}\right] - 2x_0 + \lambda(\Delta t)\,(1 - x_1^2)\,(x_1 - x_0), \tag{2.6}$$

$$\begin{aligned} x_n = x_{n-1} \left[3 - \frac{(\Delta t)^2}{2}\right] + 2(-1)^{n-1} x_0 - 4 \sum_{j=2}^{n-1} [(-1)^j x_{n-j}] \\ + (\Delta t)\,\lambda(1 - x_{n-1}^2) \left\{ x_{n-1} + (-1)^{n-1} x_0 - 2 \sum_{j=2}^{n-1} [(-1)^j x_{n-j}] \right\}, \\ n > 2. \end{aligned} \tag{2.7}$$

The method now proceeds as follows. Let $x_0^{(k)} = k + 1$, $k = 0, 1, \ldots, 10$. To each such x_0, generate, in order, for $k = 0, 1, \ldots, 10$, sequence $x_n^{(k)}$ from (2.5)–(2.7). Terminate each sequence when

$$x_{N+1}^{(k)} \geqslant x_N^{(k)} \tag{2.8}$$

and record

$$S_k = x_0^{(k)} + x_N^{(k)}. \tag{2.9}$$

Of course, in (2.8)–(2.9), N depends on k. Let $k = \mu$ be the first value of k for which

$$S_\mu \cdot S_{\mu+1} \leqslant 0. \tag{2.10}$$

Then set $a = \times x_0^{(\mu)}$ and $T/2 = N\Delta t$. Thus, $\times x_0^{(\mu)}$ is an integer which approximates a. To compute a one decimal place refinement of this approximation, set

$$x_0^{(0)} = \times x_0^{(\mu)} + 0.0, \qquad x_0^{(1)} = \times x_0^{(\mu)} + 0.1, \qquad x_0^{(1)} = \times x_0^{(\mu)} + 0.2, \ldots,$$

$$x_0^{(10)} = \times x_0^{(\mu)} + 1.0,$$

and recycle. Thus, if one had found $\times x_0^{(\mu)} = 2$, one would recycle with $x_0^{(0)} = 2.0$, $x_0^{(1)} = 2.1$, $x_0^{(2)} = 2.2, \ldots, x_0^{(10)} = 3.0$. From the resulting one decimal place refinement, one can, in the indicated fashion, construct a two decimal place refinement, and, in the same manner, eventually a j-decimal place refinement, where the magnitude of the integer j is limited only by one's computer capability.

On the UNIVAC 1108, the following approximations for a and $T/2$ were generated with $\Delta t = 0.001$ by the method of this section:

$$\begin{aligned} \lambda &= 0.1, \quad a = 2.005, \quad T/2 = 3.144, \\ \lambda &= 1.0, \quad a = 2.009, \quad T/2 = 3.332, \\ \lambda &= 10.0, \quad a = 2.014, \quad T/2 = 9.540. \end{aligned}$$

The graphs of the approximate periodic functions are shown in Fig. 1. All computations were done in double precision and the total computing time was under 20 minutes.

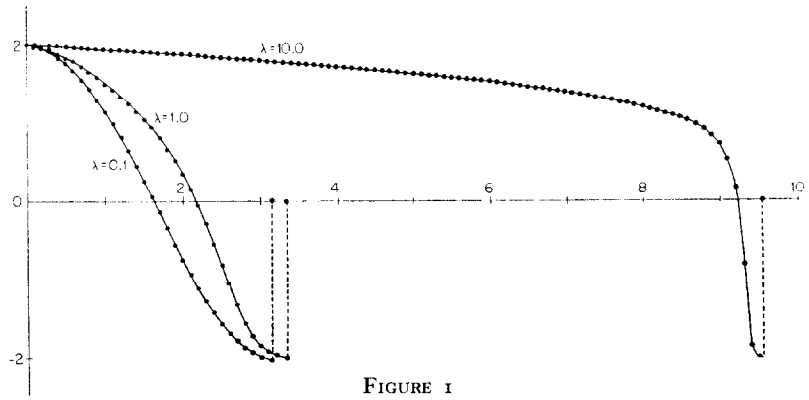

FIGURE 1

3. METHOD II: HIGH-ORDER TAYLOR SERIES

Since no single-step, Runge–Kutta formulas are known which are of order greater than seven, we will show now how easily one can construct and apply, for the problem at hand, a classical Taylor series approximation of arbitrary order greater than seven. For illustrative purposes we will concentrate on order eight.

Let $h = \Delta t$. Then, under the usual assumptions for expressing a function in a Taylor expansion, one has the eighth-order approximations:

$$\begin{aligned} x_{k+1} = x_k &+ \frac{h}{1}\dot{x}_k + \frac{h^2}{2!}\ddot{x}_k + \frac{h^3}{3!}\dddot{x}_k + \frac{h^4}{4!}x_k^{\mathrm{iv}} + \frac{h^5}{5!}x_k^{\mathrm{v}} + \frac{h^6}{6!}x_k^{\mathrm{vi}} \\ &+ \frac{h^7}{7!}x_k^{\mathrm{vii}} + \frac{h^8}{8!}x_k^{\mathrm{viii}}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \dot{x}_{k+1} = \dot{x}_k &+ \frac{h}{1}\ddot{x}_k + \frac{h^2}{2!}\dddot{x}_k + \frac{h^3}{3!}x_k^{\mathrm{iv}} + \frac{h^4}{4!}x_k^{\mathrm{v}} + \frac{h^5}{5!}x_k^{\mathrm{vi}} + \frac{h^6}{6!}x_k^{\mathrm{vii}} \\ &+ \frac{h^7}{7!}x_k^{\mathrm{viii}} + \frac{h^8}{8!}x_k^{\mathrm{ix}}, \end{aligned} \tag{3.2}$$

where, from (1.1),

$$\ddot{x}_k = \lambda \dot{x}_k - \lambda x_k{}^2 \dot{x}_k - x_k , \tag{3.3}$$

$$\dddot{x}_k = \lambda \ddot{x}_k - 2\lambda x_k (\dot{x}_k)^2 - \lambda x_k{}^2 \ddot{x}_k - \dot{x}_k , \tag{3.4}$$

$$x_k^{\mathrm{iv}} = \lambda \dddot{x}_k - 2\lambda (\dot{x}_k)^3 - 6\lambda x_k \dot{x}_k \ddot{x}_k - \lambda x_k{}^2 \dddot{x} - \ddot{x}_k , \tag{3.5}$$

$$x_k^{\mathrm{v}} = \lambda x_k^{\mathrm{iv}} - 12\lambda (\dot{x}_k)^2 \ddot{x}_k - 6\lambda x_k (\ddot{x}_k)^2 - 8\lambda x_k \dot{x}_k \dddot{x}_k - \lambda x_k{}^2 x_k^{\mathrm{iv}} - \dddot{x}_k , \tag{3.6}$$

$$\begin{aligned} x_k^{\mathrm{vi}} = {} & \lambda x_k^{\mathrm{v}} - 30\lambda \dot{x}_k (\ddot{x}_k)^2 - 20\lambda (\dot{x}_k)^2 \dddot{x}_k - 20\lambda x_k \ddot{x}_k \dddot{x}_k - 10\lambda x_k \dot{x}_k x_k^{\mathrm{iv}} \\ & - \lambda x_k{}^2 x_k^{\mathrm{v}} - x_k^{\mathrm{iv}}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} x_k^{\mathrm{vii}} = {} & \lambda x_k^{\mathrm{vi}} - 30\lambda (\ddot{x}_k)^3 - 120\lambda \dot{x}_k \ddot{x}_k \dddot{x}_k - 30\lambda (\dot{x}_k)^2 x_k^{\mathrm{iv}} - 20\lambda x_k (\dddot{x}_k)^2 \\ & - 30\lambda x_k \ddot{x}_k x_k^{\mathrm{iv}} - 12\lambda x_k \dot{x}_k x_k^{\mathrm{v}} - \lambda x_k{}^2 x_k^{\mathrm{vi}} - x_k^{\mathrm{v}} , \end{aligned} \tag{3.8}$$

$$\begin{aligned} x_k^{\mathrm{viii}} = {} & \lambda x_k^{\mathrm{vii}} - 210\lambda (\ddot{x}_k)^2 \dddot{x}_k - 140\lambda \dot{x}_k (\dddot{x}_k)^2 - 210\lambda \dot{x}_k \ddot{x}_k x_k^{\mathrm{iv}} - 42\lambda (\dot{x}_k)^2 x_k^{\mathrm{v}} \\ & - 42\lambda x_k \ddot{x}_k x_k^{\mathrm{v}} - 70\lambda x_k \dddot{x}_k x_k^{\mathrm{iv}} - 14\lambda x_k \dot{x}_k x_k^{\mathrm{vi}} - \lambda x_k{}^2 x_k^{\mathrm{vii}} - x_k^{\mathrm{vi}}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} x^{\mathrm{ix}} = {} & \lambda x_k^{\mathrm{viii}} - 560\lambda \ddot{x}_k (\dddot{x}_k)^2 - 420\lambda (\ddot{x}_k)^2 x_k^{\mathrm{iv}} - 560\lambda \dot{x}_k \dddot{x}_k x_k^{\mathrm{iv}} \\ & - 336\lambda \dot{x}_k \ddot{x}_k x_k^{\mathrm{v}} - 56\lambda (\dot{x}_k)^2 x_k^{\mathrm{vi}} - 112\lambda x_k \dddot{x}_k x_k^{\mathrm{v}} - 56\lambda x_k \ddot{x}_k x_k^{\mathrm{vi}} \\ & - 70\lambda x_k (x_k^{\mathrm{iv}})^2 - 16\lambda x_k \dot{x}_k x_k^{\mathrm{vii}} - \lambda x_k{}^2 x_k^{\mathrm{viii}} - x_k^{\mathrm{vii}}. \end{aligned} \tag{3.10}$$

The numerical method to be applied to (1.1)–(1.2) is then the same as that described in Section 2 except that the sequence x_n, $n = 1, 2, \ldots$ is no longer generated by (2.5)–(2.7) but, instead, by (3.1)–(3.10). Typical resulting computations on the UNIVAC 1108 with $\Delta t = 0.001$ yielded

$$\begin{aligned} &\lambda = 0.1, \qquad a = 2.000, \qquad T/2 = 3.148, \\ &\lambda = 1.0, \qquad a = 2.009, \qquad T/2 = 3.335, \\ &\lambda = 10.0, \qquad a = 2.014, \qquad T/2 = 9.548. \end{aligned}$$

The graphs of the resulting approximations were entirely analogous in general structure to those given in Fig. 1. All computations were in double precision and the total computing time was under 28 minutes.

It is interesting to note that packaged programs of the Taylor series method are now becoming available in which the computer determines the coefficients, the step size, and an error bound at each step of the computation in response to a given initial-value problem and a desired accuracy [1, 8].

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